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Asymmetric Collision of Gravitational Plane Waves: A New Class of Exact Solutions[1]

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A new three-parameter class of solutions to the Einstein vacuum equations is presented which represents the collision of a pair of gravitational plane waves. Depending on the choice of the parameters, one of the colliding waves has a smooth or unbounded wavefront, or it is a shock, or impulsive, or shock accompanied by an impulsive wave, while the second is any of the above types. A subfamily of the solutions develops no curvature singularity in the interaction region formed by the colliding waves.

1. INTRODUCTION

Thanks to the development of new techniques for integrating the corresponding field equations, several new analytic models representing the collision of gravitational plane waves have been added recently to the original collection which we owe to Szekeres [1, 2], Khan and Penrose [3], and Nutku and Halil [4]. (For a recent review, see Ref. [5].). One such method was developed by Chandrasekhar and Ferrari [6] and relies on the similarity of space-times admitting two space-like Killing vectors to those admitting one space-like and one time-like Killing vector field. Another solution generating technique, not less fertile than the previous one, is due to Belinsky and Zakharov [7, 8] and is usually referred to as the method of solitons (for a very lucid comparative analysis of the above techniques, see Ref. [9]).

The new models constructed by the two methods mentioned above

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brought to the surface a surprising result regarding the outcome of the collision of a pair of plane waves: The development of a space-like singularity a finite time after the instant of collision is not a generic feature of the process, as the early models had led us to believe. The first solution to show atypical behavior was obtained by Chandrasekhar and Xanthopoulos [10]. Using the Chandrasekhar–Ferrari technique, the above authors constructed a model representing the collision of variable polarization shock waves accompanied by impulsive waves in which the usual space-like singularity sealing off the future of the interaction region does not appear. Instead, the solution can be analytically extended across this hypersurface, and the singularity that eventually develops in the model turns out to be time-like.

The same type of behavior was found to characterize a subset of the class of solutions constructed by Ferrari and Ibañez [9] using the Belinsky–Zakharov soliton technique. The Ferrari–Ibañez solutions also represent the collision of a pair of shock waves accompanied by impulsive waves but, in contrast with the Chandrasekhar–Xanthopoulos model, the approaching waves have constant and parallel polarization.

In this paper a new three-parameter class of colliding plane waves solutions is presented. It is a two-parameter generalization of the Ferrari–Ibañez class of solutions mentioned above and, in general, represents the asymmetric collision of a pair of gravitational waves with constant and parallel polarization—one of which has a smooth or unbounded wavefront or is a shock or impulsive wave, or a shock wave accompanied by an impulsive one, and the other is a plane wave of any of the previous types. It is found that, for almost all of the above kinds of collision, there is a continuous range of the parameters for which the curvature singularity on the "focusing hypersurface" is avoided; therefore, the corresponding solutions can be analytically extended beyond this surface. This result offers concrete support for the new picture that is now emerging with regard to the singular behavior of colliding wave solutions. Further support in this direction comes from the analysis of the behavior of the most general solution describing collision of plane gravitational waves with constant polarization carried out very recently by Feinstein and Ibañez [11] and communicated to us while this manuscript was being prepared.

The structure of the paper is as follows. In Section 2 we use the Chandrasekhar–Ferrari gauge to obtain the solution in the interaction region formed after the collision of the waves. Using the Khan–Penrose technique, the solution is extended toward the past of the interaction region in Section 3. In Section 4, the Weyl scalars are computed, whereby the nature, as well as the singularity behavior, of the waves under collision is determined.

2. THE FIELD EQUATIONS AND THEIR SOLUTION

In a coordinate system adapted to the pair of Killing vector fields admitted by space-times representing the interaction of gravitational plane waves, the metric can be written in the form [6]

$$ds^2 = e^{\psi} \sqrt{\Delta}\, [\Delta^{-1}(d\eta)^2 - \delta^{-1}(d\mu)^2] - \sqrt{\delta\Delta}\, [\chi(dx^2)^2 + \chi^{-1}(dx^3 - q_2\, dx^2)^2] \tag{1}$$

where

$$\Delta = 1 - \eta^2 \qquad \delta = 1 - \mu^2 \tag{2}$$

and ψ, χ, and q_2 are functions of η and μ only. The time-like coordinate $\eta \in [0, 1)$ and measures time from the instant of collision, while $\mu \in (-1, 1)$ and measures the normal distance from the (x^2, x^3) planes spanned by the Killing vector fields.

When the approaching waves have aligned constant polarization, one can set

$$q_2 = 0 \tag{3}$$

globally. Then, the Einstein vacuum equations reduce to [6]

$$\delta^{-1}\mu\psi_{,\eta} + \Delta^{-1}\eta\psi_{,\mu} = -\chi^{-2}\chi_{,\eta}\chi_{,\mu} \tag{4}$$

$$2\eta\psi_{,\eta} + 2\mu\psi_{,\mu} = 3\Delta^{-1} + \delta^{-1} - \chi^{-2}[\Delta(\chi_{,\eta})^2 + \delta(\chi_{,\mu})^2] \tag{5}$$

and

$$\chi[(\Delta\chi_{,\eta})_{,\eta} - (\delta\chi_{,\mu})_{,\mu}] = \Delta(\chi_{,\eta})^2 - \delta(\chi_{,\mu})^2 \tag{6}$$

where $(\ )_{,x} \equiv \partial(\ )/\partial x$.

Introducing the function V via the equation

$$V = \ln \chi \tag{7}$$

we can write equation (6) in the form

$$(\Delta V_{,\eta})_{,\eta} - (\delta V_{,\mu})_{,\mu} = 0 \tag{8}$$

An obvious solution of Eq. (8) is given by [12]

$$V_1 = a \ln \delta\Delta = a \ln(1 - \eta^2)(1 - \mu^2) \tag{9}$$

where a is an arbitrary constant. Separation of variables, on the other hand, leads to the solution [5]

$$V_2 = \sum_{l=0}^{\infty} [A_l P_l(\eta) P_l(\mu) + B_l P_l(\eta) Q_l(\mu) + C_l Q_l(\eta) P_l(\mu) + D_l Q_l(\eta) Q_l(\mu)] \tag{10}$$

In the last expression A_l, B_l, C_l, and D_l are arbitrary constants, while P_l and Q_l are the Legendre functions of the first and second kind, respectively.

As noted by Griffiths [5], the function V corresponding to the well-known colliding wave solutions of Szekeres [1, 2] and Khan and Penrose [3] has the form (10). This is also the case with a class of solutions obtained recently by Ferrari and Ibañez [13], while in a second class of solutions obtained by the same authors V has the form $V = V_1 + V_2$. Specifically, in the latter of the Ferrari–Ibañez solutions [9]

$$V = a \ln(1-\eta^2)(1-\mu^2) - 2Q_0(\eta) P_0(\mu) \tag{11}$$

On the basis of Griffiths' observation, we have chosen to consider the case where, in the interaction region,

$$\begin{aligned} V &= a \ln(1-\eta^2)(1-\mu^2) - 2\delta_1 Q_0(\eta) P_0(\mu) - 2\delta_2 P_0(\eta) Q_0(\mu) \\ &= a \ln(1-\eta^2)(1-\mu^2) + \delta_1 \ln \frac{1-\eta}{1+\eta} + \delta_2 \ln \frac{1-\mu}{1+\mu} \end{aligned} \tag{12}$$

with δ_1, δ_2 arbitrary constants. What makes this generalization of the Ferrari–Ibañez solutions interesting is the fact that it covers the physically rich family of the Szekeres metrics as well, since the latter is obtained by letting $a = 0$ in (12).

In order to complete the solution of the field equations in the interaction region, let us first write the function ψ in the form

$$\psi = \ln[(\eta^2 - \mu^2)/\Delta^{3/4}\delta^{1/4}] + \Sigma \tag{13}$$

Inserting this expression into Eqs. (4) and (5), and taking into account (7), we obtain

$$\delta^{-1}\mu\Sigma_{,\eta} + \Delta^{-1}\eta\Sigma_{,\mu} = -V_{,\eta} V_{,\mu} \tag{14}$$

and

$$2\eta\Sigma_{,\eta} + 2\mu\Sigma_{,\mu} = -\Delta(V_{,\eta})^2 - \delta(V_{,\mu})^2 \tag{15}$$

respectively. With V given by Eq. (12), the system (14) and (15) is easily solved to give

$$\begin{aligned}\Sigma = & -(\delta_1+\delta_2)^2 \ln(\eta+\mu) - (\delta_1-\delta_2)^2 \ln(\eta-\mu) \\ & + (a+\delta_1)^2 \ln(1-\eta) + (a-\delta_1)^2 \ln(1+\eta) \\ & + (a+\delta_2)^2 \ln(1-\mu) + (a-\delta_2)^2 \ln(1+\mu) + \ln C_0 \end{aligned} \tag{16}$$

where C_0 is an arbitrary constant.

Gathering the above results, we conclude that the metric in the interaction region is given by Eq. (1) with

$$\begin{aligned} e^{\psi} &= C_0(\eta+\mu)^{\alpha}(\eta-\mu)^{\beta}\frac{(1-\eta)^{\rho}(1-\mu)^{\sigma}}{(1+\eta)^{r}(1+\mu)^{s}} \\ \chi &= \left(\frac{1-\eta}{1+\eta}\right)^{\delta_1}\left(\frac{1-\mu}{1+\mu}\right)^{\delta_2}[(1-\eta^2)(1-\mu^2)]^{a} \\ q_2 &= 0 \end{aligned} \tag{17}$$

where

$$\begin{aligned} &\alpha = 1-(\delta_1+\delta_2)^2, \qquad \beta = 1-(\delta_1-\delta_2)^2, \qquad \rho = (a+\delta_1)^2 - 3/4, \\ &\sigma = (a+\delta_2)^2 - 1/4, \qquad r = 3/4-(a-\delta_1)^2, \qquad s = 1/4-(a-\delta_2)^2 \end{aligned} \tag{18}$$

A form of the metric which is convenient for the following discussion can be obtained by the coordinate transformation

$$\eta = u\sqrt{1-v^2} + v\sqrt{1-u^2}, \qquad \mu = u\sqrt{1-v^2} - v\sqrt{1-u^2}, \qquad u, v > 0 \tag{19}$$

It reads

$$\begin{aligned} ds^2 = a_0^2 & \frac{(1-u^2)^{1/2}(1-v^2)^{1/2} - uv}{(1-v^2)^{(1-\alpha)/2}(1-u^2)^{(1-\beta)/2}} \\ & \times \frac{(1-\eta)^{\rho}(1-\mu)^{\sigma}}{(1+\eta)^{r}(1+\mu)^{s}} u^{\alpha} v^{\beta}\, du\, dv \\ & -(1-u^2-v^2)[(\chi(dx^2)^2 + \chi^{-1}(dx^3)^2] \end{aligned} \tag{20}$$

where

$$\chi = (1-u^2-v^2)^{2a}\left(\frac{1-\eta}{1+\eta}\right)^{\delta_1}\left(\frac{1-\mu}{1+\mu}\right)^{\delta_2} \tag{21}$$

$a_0^2 = 2^{\alpha+\beta+2}C_0$, and η and μ are expressed in terms of the null coordinates u and v as in Eq. (19).

3. EXTENSION OF THE SOLUTION

Equations (20) and (21) determine the space-time metric only in the interaction region where $u > 0$, $v > 0$, and $u^2 + v^2 < 1$. In order to obtain the metric in the past of this region, one usually follows Khan and Penrose [3] in letting

$$u \to uH(u) \qquad \text{and} \qquad v \to vH(v) \tag{22}$$

in the metric coefficients obtained in the region of interaction. Here, H denotes the Heaviside unit step function.

In our case, however, the direct application of the Khan–Penrose algorithm is hindered by the presence of the factors u^α, v^β in g_{uv}, unless $\alpha = \beta = 0$. But this difficulty is easily overcome by letting

$$u = \tilde{u}^n \qquad \text{and} \qquad v = \tilde{v}^m \tag{23}$$

and choosing n and m to positive and equal to

$$n = (1 + \alpha)^{-1} \qquad m = (1 + \beta)^{-1} \tag{24}$$

Indeed, the substitution of Eq. (23) into (20) and (21) leads to the expression

$$ds^2 = 2e^{-M}\, du\, dv - e^{-U}[e^V (dx^2)^2 + e^{-V}(dx^3)^2] \tag{25}$$

where

$$\begin{aligned} e^{-M} &= \frac{(1-u^{2n})^{1/2}\,(1-v^{2m})^{1/2} - u^n v^m}{(1-u^{2n})^{1-1/2m}\,(1-v^{2m})^{1-1/2n}}\,\frac{(1-\eta)^\rho\,(1-\mu)^\sigma}{(1+\eta)^r\,(1+\mu)^s} \\ e^{-U} &= 1 - u^{2n} - v^{2m} \\ e^V &= (1 - u^{2n} - v^{2m})^{2a} \left(\frac{1-\eta}{1+\eta}\right)^{\delta_1} \left(\frac{1-\mu}{1+\mu}\right)^{\delta_2} \end{aligned} \tag{26}$$

and η, μ are given by

$$\eta = u^n \sqrt{1 - v^{2m}} + v^m \sqrt{1 - u^{2n}}, \quad \mu = u^n \sqrt{1 - v^{2m}} - v^m \sqrt{1 - u^{2n}} \tag{27}$$

In writing Eqs. (25)–(27) we dropped the tildes from $\tilde{u}$ and $\tilde{v}$ for convenience in notation and used the arbitrariness of a_0 to set $mna^2{}_0 = 2$.

With the metric in the interaction region in the form given by Eqs. (25)–(27), the Khan–Penrose extension technique can be immediately applied and the result reads as follows.

In Region I, where $u<0$ and $v<0$,

$$ds^2 = 2du\, dv - (dx^2)^2 - (dx^3)^2 \tag{28}$$

In Region II, where $u<0$ and $0<v<1$, the metric has the form (25) with

$$\begin{aligned} e^{-M} &= (1+v^m)^{\sigma - r + (1-n)/2n} (1-v^m)^{\rho - s + (1-n)/2n} \\ e^{-U} &= 1 - v^{2m} \\ e^{V} &= (1+v^m)^{\delta_2 - \delta_1 + 2a} (1-v^m)^{\delta_1 - \delta_2 + 2a} \end{aligned} \tag{29}$$

Finally, in Region III, where $0<u<1$ and $v<0$, the line element is given by (25) but now,

$$\begin{aligned} e^{-M} &= (1+u^n)^{-r-s+(1-m)/2m} (1-u^n)^{\rho + \sigma + (1-m)/2m} \\ e^{-U} &= 1 - u^{2n} \\ e^{V} &= (1+u^n)^{2a - \delta_1 - \delta_2} (1-u^n)^{2a + \delta_1 + \delta_2} \end{aligned} \tag{30}$$

The projection of the above regions, as well as that of the region of interaction (Region IV), in the (u, v)-plane is shown in Fig. 1.

According to Eq. (28), space-time is flat in Region I. Equations (29) and (30), on the other hand, show that, in Region II (III) the metric depends on the null coordinate v (u) only. This suggests that the extension obtained above represents a pair of gravitational waves propagating toward each other in a flat region and colliding at $(u, v) = (0, 0)$. However, before this interpretation is accepted, we must prove that the Einstein vacuum equations are indeed satisfied in Regions II and III, as well as on the hypersurfaces $u=0$ and $v=0$ which separate Regions I–IV from each other.

Consider, in this direction, the component R_{vv} of the Ricci tensor corresponding to the line element (25), where M, U, and V are functions of u and v, only. It reads

$$2R_{vv} = -2U_{,vv} + (U_{,v})^2 + (V_{,v})^2 - 2M_{,v}U_{,v} \tag{31}$$

and, in Region II, it is the only component of the Ricci tensor that does not vanish identically. Thus, in this region, the Einstein vacuum equations reduce to $R_{vv}=0$. It is, then, a matter of simple algebra to show that the last equation is indeed satisfied by the metric coefficients given by (29). In a similar fashion one proves that the Einstein vacuum equations are satisfied in Region III, also.

With regard to the hypersurfaces, $u=0$ and $v=0$, let us note that the Khan–Penrose substitutions lead, in general, to the appearance of matter distributed on them. Thus, using Eqs. (22) and (31) together with the fact that the Einstein vacuum equations are satisfied on both sides of the $v=0$ hypersurface, we find that

$$R_{vv} = -\frac{\partial U}{\partial v}\delta(v) \tag{32}$$

where $\delta(v)$ denotes the delta function of Dirac.

Therefore, the Einstein vacuum equations will not hold on the hypersurface $v=0$ if the condition

$$\left.\frac{\partial U}{\partial v}\right|_{v=0} = 0 \tag{33}$$

is not satisfied. In fact, it is easy to show that this is the only condition that must be satisfied along $v=0$ in order for this hypersurface to remain vacuus. The symmetry between the roles of u and v, on the other hand, leads directly to

$$\left.\frac{\partial U}{\partial u}\right|_{u} = 0 \tag{34}$$

as the condition for the hypersurface $u=0$ to be vacuus. Combining Eqs. (26), (33), and (34), we conclude that no matter will appear on the separation surfaces $u=0$ and $v=0$, provided $m, n > 1/2$.

However, it is easily verified that for $1/2 < m, n < 1$ the extension obtained above is only C^0. Therefore, we will impose the condition

$$m, n \geqslant 1 \tag{35}$$

so as to make the extension C^{1-} (see p. 11 of Ref. [14] for definitions) or smoother. As it will become clear in the following section, condition (35) guarantees the integrability of the Weyl scalars corresponding to the waves under collision.

4. THE BEHAVIOR OF THE WEYL SCALARS

Let x^0 and x^1 stand for the coordinates u and v, respectively. Then, relative to the metric (25), the vectors $(\boldsymbol{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}})$, where

$$l^a = e^{M/2}\delta_1^a, \quad n^a = e^{M/2}\delta_0^a, \quad m^a = (e^{U/2}/\sqrt{2})(e^{-V/2}\delta_2^a + ie^{V/2}\delta_3^a) \tag{36}$$

form a null tetrad. As shown by Szekeres [2], the components of the Weyl tensor in the above tetrad (Weyl scalars) read as follows.

$$\begin{aligned}\psi_0 &= -(e^M/2)(V_{,11} - U_{,1} V_{,1} + M_{,1} V_{,1})\\ \psi_4 &= -(e^M/2)(V_{,00} - U_{,0} V_{,0} + M_{,0} V_{,0})\\ \psi_2 &= (e^M/2)\, M_{,01}\\ \psi_1 &= \psi_3 = 0\end{aligned} \tag{37}$$

In Region I, all the Weyl scalars vanish, of course. From Eq. (29) and (37), on the other hand, it follows that in Region II the only nonvanishing Weyl scalar is ψ_0 which is given by the expression

$$\begin{aligned}\psi_0^{\rm II} = -e^{2U+M}\{&2am^2(1-4a^2)\, v^{4m-2} - 12a^2m^2(\delta_1 - \delta_2)\, v^{3m-2}\\ &- 6am(2m-1)\, v^{2m-2} - m(m-1)(\delta_1 - \delta_2)\, v^{m-2}\}\end{aligned} \tag{38}$$

Similarly, ψ_4 is the only nonvanishing component in Region III and it is given by

$$\begin{aligned}\psi_4^{\rm III} = -e^{2U+M}\{&2an^2(1-4a^2)\, u^{4n-2} - 12a^2n^2(\delta_1 + \delta_2)\, u^{3n-2}\\ &- 6an(2n-1)\, u^{2n-2} - n(n-1)(\delta_1 + \delta_2)\, u^{n-2}\}\end{aligned} \tag{39}$$

In Region IV, all three of the components ψ_0, ψ_2, and ψ_4 are nonvanishing, but the corresponding expressions in terms of u and v are too complicated to be given here explicitly. Anyhow, as far as the behavior of the Weyl scalars across $u=0$ and $v=0$ is concerned, we only need to know the $u \to 0^+$ and $v \to 0^+$ limits of ${\psi_2}^{\rm IV}$. Because, using Eq. (37), we easily find that the Weyl scalars behave as follows as we cross the null hypersurface separating region A from region B (A → B):

(i) IV → II:

$$\begin{aligned}&\psi_0 \text{ is continuous}\\ &\psi_2 = H(u)\, \psi_2^{\rm IV}\\ &\psi_4 = H(u)\, \psi_4^{\rm IV} + e^M n(\delta_1 + \delta_2)\, u^{n-1}(1 - v^{2m})^{-1/2}\, \delta(u)\end{aligned} \tag{40}$$

(ii) IV → III:

$$\begin{aligned}&\psi_0 = H(v)\, \psi_0^{\rm IV} + e^M m(\delta_1 - \delta_2)\, v^{m-1}(1 - u^{2n})^{-1/2}\, \delta(v)\\ &\psi_2 = H(v)\, \psi_2^{\rm IV}\\ &\psi_4 \text{ is continuous}\end{aligned} \tag{41}$$

(iii) II → I:

$$\psi_0 = H(v)\,\psi_0^{\mathrm{II}} + m(\delta_1 - \delta_2)\,v^{m-1}\,\delta(v) \tag{42}$$

(iv) III → I:

$$\psi_4 = H(u)\,\psi_4^{\mathrm{III}} + n(\delta_1 + \delta_2)\,u^{n-1}\,\delta(u) \tag{43}$$

As for the limiting behavior of ψ_2^{IV} mentioned earlier, we find that

$$\psi_2^{\mathrm{IV}} \xrightarrow[u \to 0^+]{} nm(\delta_1 + \delta_2)\,u^{n-1}v^{m-1}(2av^m + \delta_1 - \delta_2)(1 - v^{2m})^{-3/2}\,(e^M|_{u=0}) \tag{44}$$

and

$$\psi_2^{\mathrm{IV}} \xrightarrow[v \to 0^+]{} nm(\delta_1 - \delta_2)\,u^{n-1}v^{m-1}(2au^n + \delta_1 + \delta_2)(1 - u^{2n})^{-3/2}\,(e^M|_{v=0}) \tag{45}$$

Equations (38) and (42) show clearly that the parameter m determines the type of the wave incident from the left in Fig. 1. Specifically, by observing the behavior of ψ_0 on crossing the $v=0$ hypersurface in the direction II → I, we can distinguish the following cases.

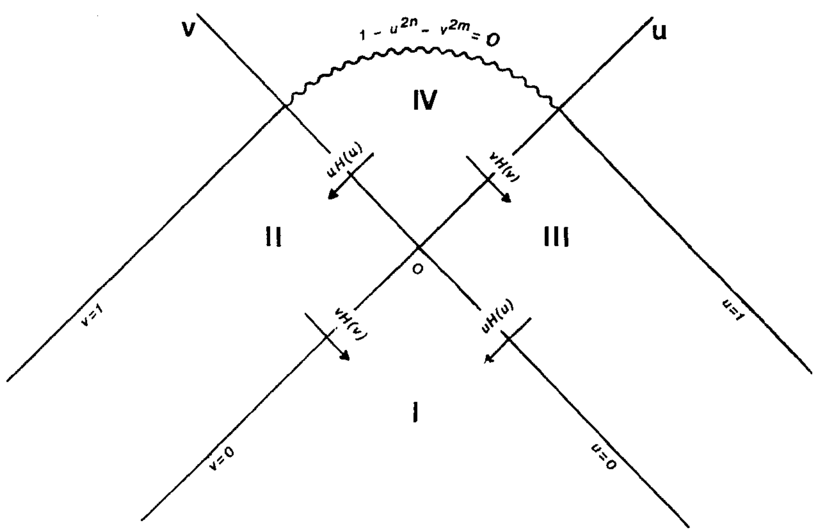

Fig. 1. The (u, v)-plane of a colliding plane waves space-time. Region I is flat. The null hypersurface $u=0$ ($v=0$) represents the wavefront of the wave incident from the left (right). The top boundary $u^{2n} + v^{2m} - 1 = 0$ of the region of interaction (IV) is the "focusing hypersurface."

(i) $m=1$. Then,

$$\psi_0 = 6aH(v) + (\delta_1 - \delta_2)\,\delta(v) \tag{46}$$

Therefore, when $a \neq 0$, the radiation incident from the left has the form of a shock wave accompanied by an impulsive one. When $a=0$, only the impulsive wave remains.

(ii) $1 < m < 2$. Then,

$$\psi_0 = m(m-1)(\delta_1 - \delta_2)\, v^{m-2} H(v) \tag{47}$$

This means that the incoming wave has an unbounded wavefront of the form $v^{-\gamma}$, with $\gamma \in (0, 1)$.

(iii) $m=2$. Then,

$$\psi_0 = 2(\delta_1 - \delta_2)\, H(v) \tag{48}$$

which corresponds to a shock wave.

(iv) $m > 2$. Then ψ_0 is continuous, with $\psi_0(0) = 0$ and the wavefront of the wave incident from the left is smooth.

By replacing v by u and m by n (and, therefore $\delta_1 - \delta_2$ by $\delta_1 + \delta_2$) in Eqs. (46)–(48), we obtain the type of the wave incident from the right in Fig. 1. It is, then, obvious that, by choosing the values of the parameters a, m, and n appropriately, we can obtain a variety of situations, whereby an impulsive wave riding a shock wave collides with a smooth wave, a shock wave collides with an impulsive or shock wave, e.t.c.

So far, we have studied the behavior of the Weyl scalars only on part of the boundary of Region IV, namely on the hypersurfaces $u=0$ and $v=0$. As far as the interpretation of the solution in terms of colliding waves is concerned, the above analysis is sufficient. However, in order to determine the end result of the collision process, one needs to know the behavior of the Weyl scalars on the rest of the boundary of Region IV, namely on the hypersurface $u^{2n} + v^{2m} = 1$.

Returning to the coordinate system of Section 2 and using Eqs. (1), (2), and (17), we find that

$$\det[g_{ab}(\eta, \mu)] = -\Delta e^{2\psi} = -C_0^2 \frac{(1-\eta)^{2\rho+1}\,(1-\mu)^{2r}}{(1+\eta)^{2r-1}\,(1+\mu)^{2s}} (\eta+\mu)^{2\alpha}\,(\eta-\mu)^{2\beta} \tag{49}$$

therefore, as $\eta \to 1$ the metric becomes singular, unless

$$2\rho + 1 = 0 \tag{50}$$

which is equivalent to

$$(a+\delta_1)^2 = 1/4 \tag{51}$$

as follows from Eq. (18). Detailed calculations show that (50) is also a sufficient condition for the Weyl scalars in Region IV to remain bounded as $\eta \to 1$. Specifically, when condition (51) holds, we find that the Weyl scalars in the region of interaction obtain the following limiting values, as $t \equiv 1 - u^{2n} - v^{2m} \to 0^+$, which according to Eq. (27), corresponds to $\eta \to 1$.

$$\begin{aligned}
\psi_2^{\mathrm{IV}} &\to 2^{r+s-\sigma-5/2} nm \frac{u^{2(ns-1)}}{v^{2(m\sigma+1)}} \left[\frac{1}{4} - (a-\delta_2)^2 - 4au^{2n}(\delta_2 + \delta_1 v^{2m}) \right] \\
\psi_0^{\mathrm{IV}} &\to -(3/2)(a+\delta_1)^{-1}\,(m/n)\,u^{1-2n} v^{2m-1} \psi_2^{\mathrm{IV}} \\
\psi_4^{\mathrm{IV}} &\to -(3/2)(a+\delta_1)^{-1}\,(n/m)\,u^{2n-1} v^{1-2m} \psi_2^{\mathrm{IV}}
\end{aligned} \tag{52}$$

Thus, when Eq. (51) (compare with Eq. (14) of Ref. [11]) is satisfied no curvature singularity develops on the "focusing hypersurface" $\eta = 1 (t=0)$. In this case the metric is extendible toward the future of the region of interaction, as well as toward its past. The specific form of the future extension of the subclass of models in which condition (51) holds will be considered in a separate article. Here, we restrict ourselves to pointing out that all of the extendible models become Petrov type D as one approaches the focusing hypersurface from the interior of Region IV. This follows from the theorem proved in Ref. [10] and the fact that, according to Eq. (52),

$$9(\psi_2^{\mathrm{IV}})^2 \to \psi_0^{\mathrm{IV}} \psi_4^{\mathrm{IV}} \tag{53}$$

as $t \to 0^+$. This is also the case (i.e., space-time is of Petrov type D) with the extendible colliding plane wave models obtained previously by Chandrasekhar and Xanthopoulos [10] and Ferrari and Ibañez [9].

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